Simulation Analyses of Behaviours of Spatially Extended Predator-Prey Systems with Random Fluctuations

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***Abstract*—We often observe some kind or another of random fluctuations in physical, chemical and social phenomena to a greater or lesser extent. The analysis of influence of such fluctuations on phenomena is very important as a basic problem in various fields including design and planning of controlled systems in control engineering and analysis of option pricing in economics. In this paper, focusing on biological communities, we study the influence of the random fluctuations on predator-prey systems with diffusion. Noting that interaction of phytoplankton and zooplankton is the basis of a food chain in the lake and the ocean, we consider the two-species predator-prey systems consists of phytoplankton and zooplankton. We analyze the influence of the random fluctuations on the spatio-temporal patterns generated by phytoplankton and zooplankton by the numerical simulations.**

***Index Terms*—Spatio-temporal pattern, predator-prey systems, stochastic reaction diffusion systems, plankton dynamics, numerical simulations**

I. INTRODUCTION

It is well known that many spatio-temporal patterns are observed in various fields of engineering and the analysis of such patterns has been one of challenging problems in pattern formations. So far, many excellent works have been done in analysis of the spatio-temporal pattern formations [1]-[6] however, most of such works are deterministic analyses. Although the deterministic analysis is effective for real phenomena, there exist some cases that the deterministic approach is not applicable. For example, random fluctuations have a great influence on the Turing structure and we often experience that the random fluctuations in the natural world change the ecological situations. In this way, analysis of the influence of random fluctuations on the phenomena is very important [7]. Among many phenomena, we especially consider the formation of the spatio-temporal patterns in biological systems [8] and we study the influence of the random fluctuations on the spatio-temporal patterns generated by two- species predator-prey system with diffusion. Taking into account the fact that phytoplankton and zooplankton construct the basic role in the food chain in the ocean, we study the two-species predator-prey system consists of such plankton.

II. STOCHASTIC PREDATOR-PREY SYSTEMS

We begin with a description of the conventional deterministic one-dimensional two-species predator-prey model introduced in $[9]-[12]$, , which is given by the reaction diffusion equation:

$$\frac{\partial u(t,x)}{\partial t} = d_u \frac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x), v(t,x)), \tag{1}$$

$$\frac{\partial v(t,x)}{\partial t} = d_v \frac{\partial^2 v(t,x)}{\partial x^2} + g(u(t,x), v(t,x)), \tag{2}$$

where $u(t,x)$ and $v(t,x)$ are densities of prey and predator at time $t \in \Theta = (0, T)$ and position $x \in G = (0, L)$, d_u and d_v are their diffusion coefficients. The functions *f(u,v)* and g*(u,v)* are related to the growth and mortality of the prey and predation by the predator. The function forms of *f(u,v)* and *g(u,v)* depend on the characteristics of the prey and the predator under consideration. In the case where two-species predator-prey system consists of phytoplankton and zooplankton, *f(u,v)* and *g(u,v)* are given in such a way that

$$f(u,v) = a\,u(1-\frac{u}{b}) - g\,\frac{u}{u+h}v, \tag{3}$$

$$g(u,v) = kg\,\frac{u}{u+h}v - mv, \tag{4}$$

where all parameters in Eqs.(3) and (4) are positive constants, a, b, h, k and m denote the maximum per capita growth rate of the prey, the carrying capacity for the prey population, the half-saturation density of the prey, the coefficient of food utilization and the mortality of the predator.

The 1st term of the R.H.S. of Eq.(3) means the local growth and natural mortality of the prey, which is given by the logistic growth. The 2nd term of the R.H.S. of Eq.(3) and the 1st term of the R.H.S. of Eq.(4) denote the trophical interaction of the prey and the predator, which is given by the so-called Holling type II trophical response [9] – [12]. In [9]

– [12] assuming that the diffusion coefficients d_u and d_v are equal, the analysis is performed. This assumption is justified from the fact that the mixing is mainly caused by marine turbulence in natural plankton communities and also is appropriate for consideration of the non-Turing pattern formation.

In this paper, since we are interested in the influence of the random fluctuations in plankton communities on their spatio-temporal pattern formation, instead of the deterministic model mentioned above, we propose the stochastic model below:

$$\frac{\partial u(t,x)}{\partial t} = d_u \frac{\partial^2 u(t,x)}{\partial x^2} + a\,u(t,x)(1-\frac{u(t,x)}{b}) -g\,\frac{u}{u+h} v(t,x) + r_u u(t,x)\frac{\partial^2 w(t,x)}{\partial t \partial x}, \tag{5}$$

$$\frac{\partial v(t,x)}{\partial t} = d_v \frac{\partial^2 v(t,x)}{\partial x^2} + kg\,\frac{u(t,x)}{u(t,x)+h} v(t,x) -mv(t,x) + r_v v(t,x)\frac{\partial^2 q(t,x)}{\partial t \partial x}, \tag{6}$$

where $w(t,x)$ and $q(t,x)$ are mutually inde- pendent two-parameter Wiener processes [13].

Assuming $d_u = d_v \equiv d$, according to [9] – [12], we also introduce dimensionless variables such that

$$\tilde{u} = u/b,\ \tilde{v} = vg/(a\,b),\ \tilde{t} = a\,t,\ \tilde{x} = x(a/d)^{\frac{1}{2}},$$
$$\tilde{h} = h/b,\ \tilde{r}_u = r_u(a^3/d)^{\frac{1}{4}},\ \tilde{r}_v = r_v(1/a\,d)^{\frac{1}{4}}. \tag{7}$$

It should be noted that the 0.5-self-similarity [13] of the Wiener process such that

$$w(Tt, Sx) = T^{\frac{1}{2}} S^{\frac{1}{2}} w(t,x). \tag{8}$$

The equality in Eq.(8) holds in the sense of the distribution.

It follows from Eqs.(5) to (8) that

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \tilde{u}(1-\tilde{u}) - \frac{\tilde{u}}{\tilde{u}+\tilde{h}}\tilde{v} + \tilde{r}_u \tilde{u}\frac{\partial^2 w}{\partial \tilde{t}\partial \tilde{x}}, \tag{9}$$

$$\frac{\partial \tilde{v}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + k\frac{\tilde{u}}{\tilde{u}+\tilde{h}}\tilde{v} - m\tilde{v} + \tilde{r}_v \tilde{v}\frac{\partial^2 q}{\partial \tilde{t}\partial \tilde{x}}, \tag{10}$$

where $k = kg/a$, $m = m/a$ and the superscript is omitted in the sequel for the simplicity of descriptions.

We assume that the initial and the boundary conditions are given by

$$u(0,x) = u_0(x),\quad v(0,x) = v_0(x),\quad x \in G, \tag{11}$$

$$\frac{\partial u(t,x)}{\partial n} = \frac{\partial v(t,x)}{\partial n} = 0,\quad (t,x) \in \Theta \times \partial G, \tag{12}$$

where subscripts denote partial derivatives corresponding to each subscript, $\Theta = (0,T)$ (T is a positive constant), $\partial(\cdot)/\partial n$ denotes the exterior normal derivative on the boundary ∂G of G.

The pattern formation problems in the two-species predator-prey systems without random fluctuations in Eqs.(5) and (6)(or Eqs.(9) and (10)) have already been studied by Petrovskii and Malchow [9] – [12], and they found that an irregular pattern appears in some conditions in the initial predator distribution $v_0(x)$ (or the prey distribution $u_0(x)$). Main feature of this paper is to study the influence of the random fluctuations on spatio-temporal patterns in two-species predator-prey system, which is impossible by the model in Petrovskii and Malchow [9] – [12].

III. Deterministic Linear Stability Analysis

In order to study the dominant behavior of Eqs.(9) and (10), consider the linear stability of Eqs.(9) and (10) without the random noises and diffusion, i.e.,

$$\frac{du(t)}{dt} = u(t)(1-u(t)) - \frac{u(t)}{u(t)+h}v(t), \tag{13}$$

$$\frac{dv(t)}{dt} = k\frac{u(t)}{u(t)+h}v(t) - mv(t). \tag{14}$$

It is easily shown that Eqs.(13) and (14) have three spatially uniform equilibrium states $(u^*, v^*) = (0,0),\ (1,0)$ and (u_e, v_e) (co-existence of the prey and the predator) such that

$$u_e = \frac{ph}{1-p},\ v_e = (1-u_e)(h+u_e),\quad (p = \frac{m}{k}). \tag{15}$$

Although the linear stability of the equilibrium state (u^*, v^*) depends on the values h, p and a , it follows from the linear stability analysis that the equilibrium states (0, 0) and (1, 0) are always saddle points and the equilibrium state (u_e, v_e) are unstable (stable) if

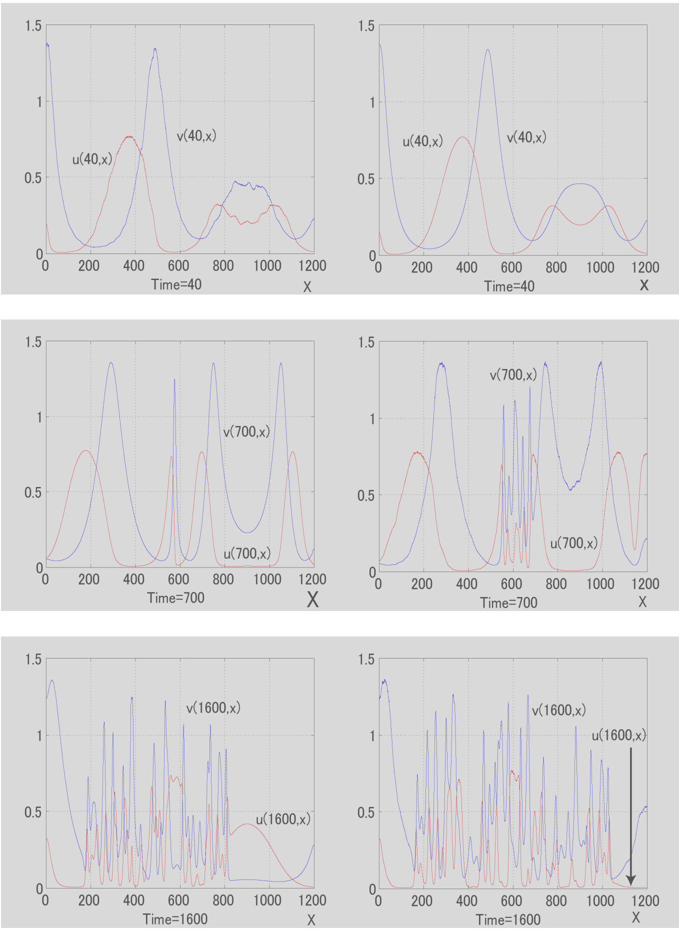
$$h < (1-p)/(1+p)\ \ (h > (1-p)/(1+p)). \tag{16}$$

IV. Simulations

In this paper, taking parameters so as for the equilibrium state (u_e, v_e) to be the unstable focus and changing the initial distribution of the prey, we study the influence of the random fluctuations on the predator-prey systems by numerical simulations. Setting parameters as $a = 2.0$, $b = 0.6$, $d_u = d_v = 1$, $h = 0.4$ and taking the noise coefficients and the spatial region G as $r_u = r_v = 0.008$ and $G = (0,\ 1200)$, under the initial densities $u_0(x) = u_e$ and

$v_0(x) = v_e + 0.0000025(x-900)^2 + d$ with two different d, simulations are performed.

(Case-1) $d = 0.005$: Results of simulations are shown in Figure 1 and Figures 1(a) and 1(b) are time evolution of densities of the prey $u(t,x)$ and the predator $v(t,x)$ under the no noise and the noise. From Figure 1, we can see that in the noise case, the irregular region appears earlier than the no noise case. In order to see this more clearly, time evolution of the density of the prey $u(t,x)$ in the (t,x)-plain is shown in Figures 2 and 3. In Figures 2 and 3, value of the density of $u(t,x)$ is given by the color indicated at the color bar. From Figures 2 and 3, in both cases of the no noise and noise, the irregular region propagates at the constant speed and the onset of the irregular pattern in the noise case is earlier than the no noise case. Figure 4 is the (u,v)-phase plain at $x = 250$ for $1000 < t < 2000$ under the no noise and the noise.



(a) no noise case (b) noise case

Figure 1. Time evolution of densities $u(t,x)$ and $v(t,x)$ of the prey and the predator under the no noise (left) and noise (right).

Since Figure 4 captures only the characteristics of the behavior of the prey and the predator at the special spatial point x= 250, in order to see the global characteristics, define

$$u_A(t) = \frac{1}{L}\int_0^L u(t,x)dx, v_A(t) = \frac{1}{L}\int_0^L v(t,x)dx. \quad (16)$$

The $(u_A(t), v_A(t))$ phase plain is depicted in Figure 5. It follows from Figures 4 and 5 that the predator and the prey randomly move in some restricted region in the phase plain in the no noise and the noise cases.

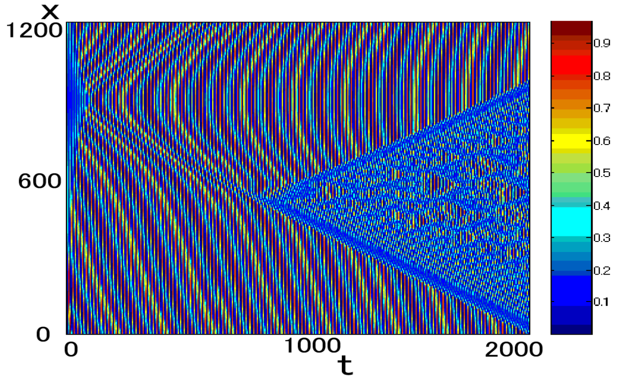


Figure 2. Behaviour of the prey density *u(t,x)* in the *(t,x)* -plain under the no noise.

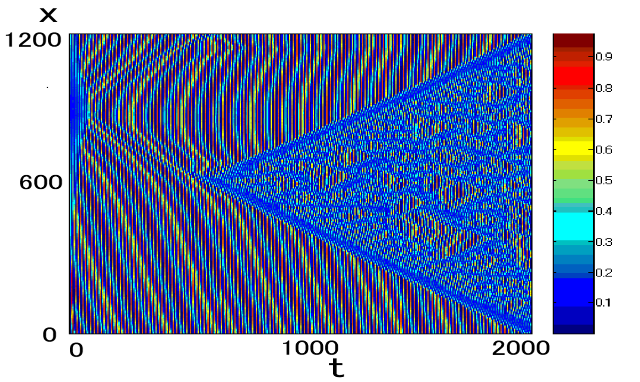
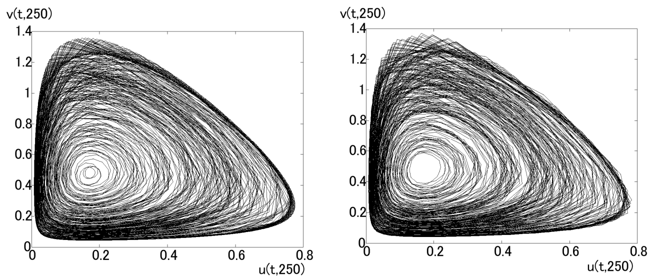
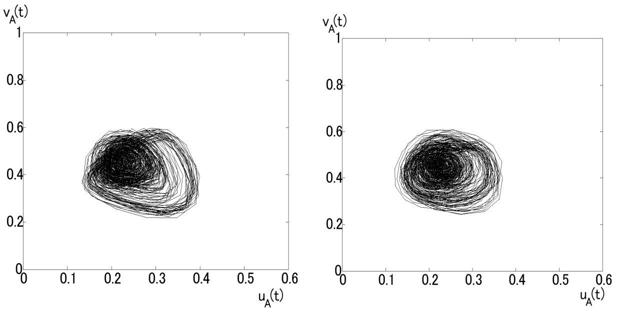


Figure 3. Behaviour of the prey density *u(t,x)* in the *(t,x)* -plain under the noise.



(a) no noise case (b) noise case

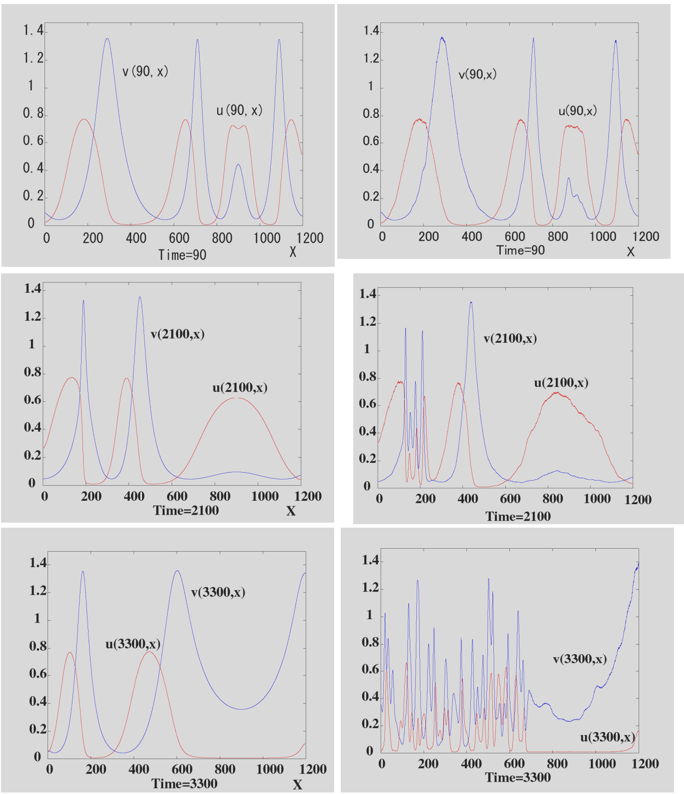
Figure 4. *(y, v)* -phase plain at *x*=250 for 1000< *t* <2000 under the no noise (left) and noise (right).



(a) no noise case (b) noise case

Figure 5. Phase plane *($u_A(t)$, $v_A(t)$)* of spatially averaged densities for 1000< *t* <2000.

(Case 2) $d = 0.015$: It should be noted that the distance between minimum value of the initial predator distribution $v_0(x)$ and the equilibrium

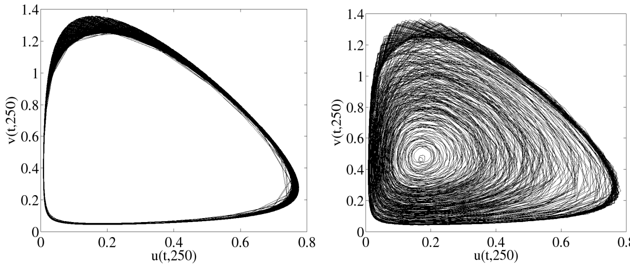


(a) no noise case (b) noise case

Figure 6. Time evolution of densities $u(t,x)$ and $v(t,x)$ of the prey and the predator under the no noise (left) and noise (right).

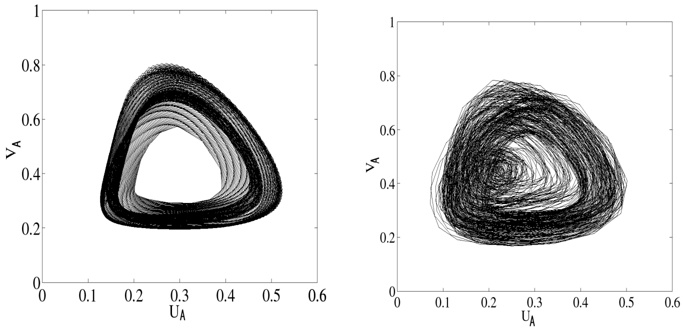
state v_e is longer than one in Case-1.

It follows from Figure 6(a) that behaviour of the densities of the prey and the predator under the no noise has some constant period so that the limit cycle is formed as shown in Figure 6(a). Figure 7 denotes the time evolution of the prey and the predator in the (u,v) -phase plane at *x*=250 for 1000 < *t* < 5000. The $(u_A(t), v_A(t))$ phase plain is depicted in Figure 8. In the noise case, the behaviour of the prey and the predator becomes very irregular as shown in Figures 5(b), 6(b) and 8(b). In this case, the prey and predator move randomly inside of the limit cycle. From Figures 9 and 10, we can see that the irregular region propagates at the constant speed in the no noise and the noise cases. In this way, random uncertainties can drastically change the behaviour of the predator and the prey. In the real biological systems, we often observe that the predator and the prey form patchy inhabitable regions. One of reason of generation of patchy region seems randomness in the environmental situations.



(a) no noise case (b) noise case

Figure 7. *(u, v)* -phase plain at *x*=250 for 1000< *t* <2000 under the no noise and noise.



(a) no noise case (b) noise case

Figure 8. Phase plane *($u_A(t)$, $v_A(t)$)* of spatially averaged densities for 1000< *t* <2000.

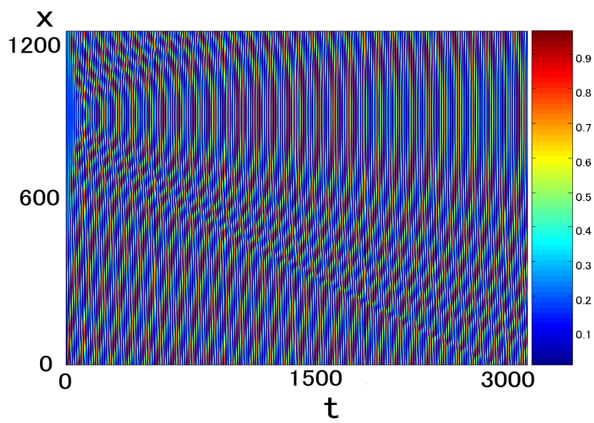


Figure 9. Behaviour of the prey density *u(t,x)* in the *(t, x)* -plain under the no noise.

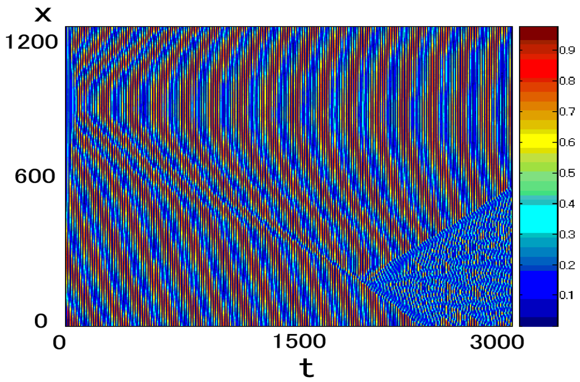


Figure 10. Behaviour of the prey density u*(t, x)* in the *(t, x)* -plain under the noise.

V. CONCLUSIONS

In this paper, the influence of the random uncertainties on the spatio-temporal pattern formation in the predator-prey systems with diffusion has been studied by numerical simulations.

From the simulation results, we can see that random uncertainties can drastically change the behavior of the predator and the prey and random uncertainties caused by the fluctuations of environmental situations seem to be part of the reason that patchy region is generated in ecosystems.

Petrovskii and Malchow [9] – [12] found that irregular region is generated in the case where the initial prey (or predator) distribution has intersection with its equilibrium state or it has a sharp inclination, however, we show that the irregular region can appear in the existence of the random fluctuations even if the conditions in Petrovskii and Malchow [9] – [12] are not satisfied.

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