APPLICATION OF THE MAXIMUM PRINCIPLE TO SINGULARLY PERTURBED SYSTEMS WITH VARIABLE RANGE OF PHASE SPACE SOLUTION

Fedir SOPRONIUK, Maxim NEGADAYLOV
Yury Fedkovich National University of Chernivtsi
str.Universietetska nr.28, UA-58012 Chernivtsi
fosopr@chnu.cv.ua

Abstract. *Developed the method for finding solve of singularly perturbed dynamic control system with variable structure, that essentially using the Pontriagin maximum principle.*
Keywords: *phase space, Pontriagin maximum principle, singularly perturbed systems, equations.*

Introduction

In the work for systems with variable range of phase space [1,2] with singularly perturbation [3,4] we propose the algorithm for finding solve. Algorithm essentially using Pontriagin maximum principle.

Mathematical model singularly perturbed system with variable range of phase space

On the segment $[T_0, T_1]$ with restricted partition $\tau = \{\tau_j,\ j = \overline{1,N}\}$, where $\tau_j = \{t : t \in [t_{j-1}, t_j)\}$, $j = 1,2,..., N-1$, $\tau_N = \{t : t \in [t_{N-1}, t_N]\}$, $t_0 = T_0 < < t_1 < ... < t_{N-1} < t_N = T_1$ let's consider the system, the dynamics of which has the next mathematical model:

$$\frac{dx_1^{(j)}(t)}{dt} = A_{11}^{(j)}(t)x_1^{(j)}(t) + A_{12}^{(j)}(t)x_2^{(j)}(t), \quad (1)$$

$$\varepsilon_j \frac{dx_2^{(j)}(t)}{dt} = A_{21}^{(j)}(t)x_1^{(j)}(t) + A_{22}^{(j)}(t)x_2^{(j)}(t), \quad (2)$$

with variable conditions of phase space range

$$x_1^{(j)}(t_{j-1}) = C_{11}^{(j)} x_1^{(j-1)}(t_{j-1} -) + C_{12}^{(j)} x_2^{(j-1)}(t_{j-1} -), \quad (3)$$

$$x_2^{(j)}(t_{j-1}) = C_{21}^{(j)} x_1^{(j-1)}(t_{j-1} -) + C_{22}^{(j)} x_2^{(j-1)}(t_{j-1} -). \quad (4)$$

In the relations (1)–(4): $x_1^{(j)}(t), x_2^{(j)}(t)$ – respectively n_1^j– measurable and n_2^j– measurable vectors of phase state if $t \in \tau_j$, $A_{11}^{(j)}(t), A_{12}^{(j)}(t), A_{21}^{(j)}(t), A_{22}^{(j)}(t)$ – known matrix, with size $n_1^j \times n_1^j$, $n_1^j \times n_2^j$, $n_2^j \times n_1^j$, $n_2^j \times n_2^j$ respectively, and matrix $A_{11}^{(j)}(t), A_{21}^{(j)}(t), A_{22}^{(j)}(t)$ have piecewise continuous elements, matrix $A_{12}^{(j)}(t)$ – differential elements under $t \in \tau_j$, $C_{11}^{(j)}, C_{12}^{(j)}, C_{21}^{(j)}, C_{22}^{(j)}$ – rectangular matrix whith size $n_1^j \times n_1^{j-1}$, $n_1^j \times n_2^{j-1}$, $n_2^j \times n_1^{j-1}$, $n_2^j \times n_2^{j-1}$ respectively, $\varepsilon_j > 0$ – small parameter, $j = \overline{1,N}$. Furthermore, we consider, that if $j = 1$ then the next equals is right: $C_{11}^{(1)} = E_1^{(1)}$, $C_{12}^{(1)} = 0$, $C_{21}^{(1)} = 0$, $C_{22}^{(1)} = E_2^{(1)}$, where $E_1^{(1)}$, $E_2^{(1)}$ – unitary matrix with orders n_1^1 and n_2^1 respectively, $C_{12}^{(1)}, C_{21}^{(1)}$ – null matrix with size $n_1^1 \times n_2^1$, $n_2^1 \times n_1^1$, $x_1^{(0)}(t_0) = x_1^{(1)}(t_0) = x_{10}^{(1)}$, $x_2^{(0)}(t_0) = x_2^{(1)}(t_0) = x_{20}^{(1)}$ – the starting phase conditions of the system (1), (2) respectively under $t = t_0$.

Assuming, that quality of functioning system (1), (2) determining by value of the functional

$$I\left(x_1^{(1)}(\cdot),..., x_1^{(N)}(\cdot), x_2^{(1)}(\cdot),..., x_2^{(N)}(\cdot)\right) =$$

$$= \frac{1}{2}\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\left(x_1^{(j)*}(s)Q_1^{(j)}(s)x_1^{(j)}(s)+x_2^{(j)*}Q_2^{(j)}(s)x_2^{(j)}(s)\right)ds+$$

$$+\frac{1}{2}x_1^{(N)*}(t_N)Q_3^{(N)}x_1^{(N)}, \qquad (5)$$

where $Q_1^{(j)}(t), Q_2^{(j)}(t), Q_3^{(N)}$ – symmetrical positive-defining matrix with sizes $n_1^j \times n_1^j$, $n_2^j \times n_2^j$, $n_1^N \times n_1^N$ respectively, matrix elements $Q_2^{(j)}(t)$ differentiable when $t \in \tau_j$, $j=\overline{1,N}$, the symbol '*' meaning the transposition operation.

Formulate of the task. Main means and confirmations.

Problem 1. Finding the minimum of the functional (5) per $x_2^{(1)}(t),...,x_2^{(N)}(t)$ under the next limitations: $x_1^{(1)}(t),...,x_1^{(N)}(t)$ is the solve of (1) with the conditions (3).
Assume that $X_1^{(j)}(t,s)$ – normal fundamention solution corresponding (1) homogeneous system, or matrix solution of the next task:

$$\frac{dX_1^{(j)}(t,s)}{dt} = A_{11}^{(j)}(t)X_1^{(j)}(t,s),$$

$$X_1^{(j)}(s,s) = E_1^{(j)}, \qquad (6)$$

where $E_1^{(j)}$ – unitary matrix with sizes $n_1^j \times n_1^j$, $s \in \tau_j$, $t \in \tau_j$, $j=\overline{1,N}$.
Than, solution (1), that satisfy starting condition $x_1^{(1)}(t_0) = x_{10}^{(1)}$ and condition (3), is

$$x_1^{(j)}(t) = X_1^{(j)}(t,t_{j-1})C_{11}^{(j)}...X_1^{(j)}(t_1,t_0)C_{11}^{(1)}x_{10}^{(1)} +$$

$$+\sum_{k=1}^{j-1}\int_{t_{k-1}}^{t_k}W_k^{(j)}(t,s)A_{12}^{(k)}(s)x_2^{(j)}(s)ds+$$

$$+\int_{t_{j-1}}^{t}W_j^{(j)}(t,s)A_{12}^{(j)}(s)x_2^{(j)}(s)ds+$$

$$+\sum_{k=1}^{j}W_k^{(j)}(t,t_k)C_{12}^{(k)}x_{20}^{(1)}, \qquad (7)$$

where

$$W_k^{(j)}(t,s) = X_1^{(j)}(t,t_{j-1})C_{11}^{(j)}X_1^{(j-1)}(t_{j-1},t_{j-2})C_{11}^{(j-1)}...$$

$$...X_1^{(k+1)}(t_{k+1},t_k)C_{11}^{(k+1)}X_1^{(k)}(t_k,s),$$

$$s \in \tau_k,\ t \in \tau_j,\ 1 \le k \le j \le N.$$

Ground of the solve construction for singularly perturbed system with variable range of phase space

Theorem. The solution of the problem 1 is

$$x_2^{(j)o}(t) = \left(Q_2^{(j)}(t)\right)^{-1}A_{12}^{(j)*}(t)R^{(j)}(t)x_1^{(j)}(t), \quad (8)$$

where $R^{(j)}(t)$ – matrix solution of the tasks

$$\frac{dR^{(j)}(t)}{dt} = -A_{11}^{(j)*}(t)R^{(j)}(t)+Q_1^{(j)}(t), \quad (9)$$

$$R^{(j)}(t_j -) = C_{11}^{(j+1)}R^{(j+1)}(t_j)C_{11}^{(j+1)},$$

$$j = N-1, N-2,...,1,$$

$$R^{(N)}(t_N) = -Q_3^{(N)}. \qquad (10)$$

The proof. Assume that $x_2^{(j)}(t)$, $t \in \tau_j$, $j=\overline{1,N}$ are control functions for the system (1) with the conditions (3), examine the next functions:

$$H\left(x_1^{(1)}(t),...,x_1^{(N)}(t),x_2^{(1)}(t),...,x_2^{(N)}(t),\psi^{(1)}(t),...,\psi^{(N)}(t),t\right)=$$

$$=-\frac{1}{2}\sum_{j=1}^{N}\left(x_1^{(j)*}(t)Q_1^{(j)}(t)x_1^{(j)}(t)+x_2^{(j)*}Q_2^{(j)}(t)x_2^{(j)}(t)\right)+$$

$$+\sum_{j=1}^{N}\psi^{(j)*}(t)\left(A_{11}^{(j)}(t)x_1^{(j)}(t)+A_{12}^{(j)}(t)x_2^{(j)}(t)\right), \quad (11)$$

$$H\left(x_1^{(j)}, x_2^{(j)}, \psi^{(j+1)}\right) = \psi^{(j+1)*}\left(t_j\right)\left(C_{11}^{(j+1)} x_1^{(j)}\left(t_j -\right) +\right.$$

$$\left. + C_{12}^{(j+1)} x_2^{(j)}\left(t_j -\right)\right), \qquad (12)$$

where $\psi^{(j)}(t)$ is the solution of adjoint system

$$\frac{d\psi^{(j)}(t)}{dt} = -grad_{x_1^{(j)}}\ H\left(x_1^{(1)}(t),\ldots,x_1^{(N)}(t), x_2^{(1)}(t),\right.$$

$$\left.\ldots, x_2^{(N)}(t), \psi^{(1)}(t),\ldots,\psi^{(N)}(t), t\right),$$

$$\psi^{(j)}\left(t_j -\right) =$$

$$= grad_{x_1^{(j)}} H\left(x_1^{(j)}\left(t_j -\right), x_2^{(j)}\left(t_j -\right), \psi^{(j+1)}\left(t_j\right)\right),$$

$$j = N-1, N-2, \ldots, 1,$$

$$\psi^{(N)}\left(t_N\right) = -Q_3^{(N)} x_1^{(N)}\left(t_N\right).$$

Apply the Pontriagin maximum principle, $x_2^{(j)o}(t)$ finding as the solution of the equation

$$-Q_2^{(j)}(t) x_2^{(j)o}(t) + A_{12}^{(j)}(t)\psi^{(j)}(t) = 0,$$

or

$$x_2^{(j)o}(t) = \left(Q_2^{(j)}(t)\right)^{-1} A_{12}^{(j)}(t)\psi^{(j)}(t),\ t \in \tau_j,$$

$$j = \overline{1,N}. \qquad (13)$$

Calculating

$$grad_{x_1^{(j)}} H\left(x_1^{(1)}(t),\ldots,x_2^{(N)}(t), \psi^{(1)}(t),\ldots,\psi^{(N)}(t), t\right)$$

finding the systems of equations for find the adjoint variables $\psi^{(j)}(t)$

$$\frac{d\psi^{(j)}(t)}{dt} = -A_{11}^{(j)*}\psi^{(j)}(t) + Q_1^{(j)}(t) x_1^{(j)}(t), \qquad (14)$$

$$\psi^{(j)}\left(t_j -\right) = C_{11}^{(j+1)*}\psi^{(j+1)}\left(t_j\right),$$

$$\psi^{(N)}\left(t_N\right) = -Q_3^{(N)} x_1^{(N)}\left(t_N\right). \qquad (15)$$

Finding $\psi^{(j)}(t)$ as

$$\psi^{(j)}(t) = R^{(j)}(t) x_1^{(j)}(t). \qquad (16)$$

where $R^{(j)}(t)$ – unknown matrix with size $n_1^j \times n_1^j$.

By substituting (16) in (14), finding equation (9) and conditions of the over patching structures (10) for finding matrixes $R^{(j)}(t)$ with $t \in \tau_j$, $j = \overline{1,N}$.

The formulas (13), (16) and conditions (4) completely defined functions $x_2^{(j)o}(t)$, substituting that in (7), finding $x_1^{(j)}(t)$ with $t \in \tau_j$ for all $j = \overline{1,N}$.

Solve of the task

$$\frac{dx_1^{(j)}(t)}{dt} = \left(A_{11}^{(j)}(t) + A_{12}^{(j)}(t)\left(Q_2^{(j)}(t)\right)^{-1} A_{12}^{(j)*}(t) R^{(j)}(t)\right)\times$$

$$\times x_1^{(j)}(t), \qquad (17)$$

$$x_1^{(j)}\left(t_{j-1}\right) = \left(C_{11}^{(j)} + C_{12}^{(j)}\left(Q_2^{(j)}\left(t_{j-1} -\right)\right)^{-1} A_{12}^{(j)*}\left(t_{j-1} -\right)\right)\times$$

$$\times R^{(j)}\left(t_{j-1} -\right) x_1^{(j-1)}\left(t_{j-1} -\right) \qquad (18)$$

finding the solve (1) $x_1^{(j)}(t)$ that satisfy the starting condition $x_1^{(1)}(t_0) = x_{10}^{(1)}$ on the assumption of functional (5) obtain minimum valuation when $x_2^{(j)}(t) = x_2^{(j)o}(t)$, $t \in \tau_j$, $j = \overline{1,N}$.

Now in the system (2) make substitute

$$x_1^{(j)}(t) = x_1^{(j)}(t), \qquad (19)$$

$$x_2^{(j)}(t) = x_2^{(j)o}(t) + z^{(j)}(t), \qquad (20)$$

where $z^{(j)}(t)$ – unknown vector-functions corresponding dimension.

So far as

$$\frac{dx_2^{(j)o}(t)}{dt} = \left(Q_2^{(j)}(t)\right)^{-1}\left(-\frac{dQ_2^{(j)}(t)}{dt}\left(Q_2^{(j)}(t)\right)^{-1}A_{12}^{(j)*}(t)R^{(j)}(t) + \right.$$

$$+\frac{dA_{12}^{(j)*}(t)}{dt}R^{(j)}(t) - A_{12}^{(j)*}(t)A_{11}^{(j)*}(t)R^{(j)}(t) + A_{12}^{(j)*}(t)Q_1^{(j)}(t) +$$

$$+A_{12}^{(j)*}(t)R^{(j)}(t)A_{11}^{(j)}(t) +$$

$$\left. + A_{12}^{(j)*}(t)R^{(j)}(t)A_{12}^{(j)}(t)\left(Q_2^{(j)}(t)\right)^{-1}A_{12}^{(j)*}(t)R^{(j)}(t)\right)\times$$

$$\times x_1^{(j)}(t), \qquad (21)$$

than, designation

$$\frac{dQ_2^{(j)}(t)}{dt} = \overline{Q}_2^{(j)}(t), \; \frac{dA_{12}^{(j)*}(t)}{dt} = \overline{A}_{12}^{(j)*}(t),$$

get the next task for the finding $z^{(j)}(t)$:

$$\varepsilon_j\frac{dz^{(j)}(t)}{dt} = A_{22}^{(j)}(t)z^{(j)}(t) +$$

$$+\left(A_{21}^{(j)}(t) - A_{22}^{(j)}(t)\left(Q_2^{(j)}(t)\right)^{-1}A_{12}^{(j)*}(t)R^{(j)}(t)\right)x_1^{(j)}(t) -$$

$$-\varepsilon_j\left(Q_2^{(j)}(t)\right)^{-1}\left(-\overline{Q}_2^{(j)}(t)\left(Q_2^{(j)}(t)\right)^{-1}A_{12}^{(j)*}(t)R^{(j)}(t) + \right.$$

$$+\overline{A}_{12}^{(j)*}(t)R^{(j)}(t) - A_{12}^{(j)*}(t)A_{11}^{(j)*}(t)R^{(j)}(t) +$$

$$+A_{12}^{(j)*}(t)Q_1^{(j)}(t) + A_{12}^{(j)*}(t)R^{(j)}(t)A_{11}^{(j)}(t) +$$

$$\left. + A_{12}^{(j)*}(t)R^{(j)}(t)A_{12}^{(j)}(t)\left(Q_2^{(j)}(t)\right)^{-1}A_{12}^{(j)*}(t)R^{(j)}(t)\right)\times$$

$$\times x_1^{(j)}(t), \qquad (22)$$

with the conditions of phase space range change

$$z^{(1)}(t_0) = x_{20}^{(1)} - \left(Q_2^{(1)}(t_0)\right)^{-1}A_{12}^{(1)*}(t_0)R^{(1)}(t_0)x_{10}^{(1)},$$

when $j = 1$, and

$$z^{(j)}(t_{j-1}) = C_{21}^{(j)}x_1^{(j-1)}(t_{j-1}-) +$$

$$+C_{22}^{(j)}\left(z^{(j-1)}(t_{j-1}-) + x_2^{(j-1)}(t_{j-1}-)\right),$$

when $j = 2,3,...,N$.

References

[1] Сопронюк Ф.О. (1995) *Моделювання та оптиці-зація систем управління з розгалуженням структур.* – Чернівці: Рута.

[2] Kirichenko M., Soproniuk F. (1996) *Structured Methods for Control Systems Research* // Development and Application Systems. – Numb.6. – Suceava (Rom).

[3] Моисеев Н.Н. (1981) *Математические задачи системного анализа.* – М.: Наука.

[4] Soproniuk Ye. (2002) *Investigation of Properties of Singularly Perturbed Control Systems with Varying Structure* // ISBN 973 – 98670 – 9 - X . Proceedings of the 6th International Conference on Development and Application Systems DAS 2002, 23 – 25 May. – Suceava, Romania.